Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Discrete Mathematics 306 (2006) 19–25

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

On integral sum graphs

Zhibo Chen

Department of Mathematics, Penn State University, McKeesport, PA 15132, USA

Received 11 April 2003; received in revised form 28 September 2005; accepted 31 October 2005
Available online 4 January 2006

Abstract

A graph G is said to be an *integral sum graph* if its nodes can be given a labeling f with distinct integers, so that for any two distinct nodes u and v of G, uv is an edge of G if and only if $f(u) + f(v) = f(w)$ for some node w in G. A node of G is called a saturated node if it is adjacent to every other node of G. We show that any integral sum graph which is not K_3 has at most two saturated nodes. We determine the structure for all integral sum graphs with exactly two saturated nodes, and give an upper bound for the number of edges of a connected integral sum graph with no saturated nodes. We introduce a method of identification on constructing new connected integral sum graphs from given integral sum graphs with a saturated node. Moreover, we show that every graph is an induced subgraph of a connected integral sum graph. Miscellaneous related results are also presented.

Keywords: Integral sum graph; Sum graph; Saturated node

1. Introduction

All graphs in this paper are finite and have no loops or multiple edges. We follow in general the graph-theoretic notation and terminology of [4] unless otherwise specified.

As introduced by Harary [6], a graph G is said to be an *integral sum graph* if its nodes can be given a labeling f with distinct integers, so that for any two distinct nodes u and v of G, uv is an edge of G if and only if $f(u) + f(v) = f(w)$ for some node w in G. (And such a labeling f is then called an *integral sum labeling* of G.) If there is an integral sum labeling f of G with $f(x) > 0$ for all nodes x in G, then G is said to be a *sum graph*. Note that the concept of sum graphs was introduced earlier in Harary [5], and much work has been devoted to sum graphs. For example, Ellingham [2] proved a conjecture of Harary that the disjoint union of a single node K_1 with any tree is a sum graph. For a survey on sum graphs and integral sum graphs, please refer to the dynamic survey on graph labeling by Gallian [3].

It is easily seen that any nontrivial graph G (i.e., G has more than one node) is not a sum graph if G is connected. However, many nontrivial connected graphs are integral sum graphs. For example, Harary [6] found that all paths and stars are integral sum graphs. Sharary [7] showed that the cycles C_n and the wheels W_n are also integral sum graphs for all $n \neq 4$. In [1] we introduced some methods on constructing new connected integral sum graphs from given integral sum graphs by identification. As applications of these methods of identification, we proved that the generalized stars

E-mail address: zxc4@psu.edu.

0012-365X/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2005.10.021

(obtained from a star by extending each edge to a path) and the trees all of whose nodes of degree not 2 are at least distance 4 apart are integral sum graphs.

In the present paper, we call a node of graph G a saturated node if it is adjacent to every other node of G. We show that all integral sum graphs except the complete graph K_3 cannot have more than two saturated nodes. We determine the structure for all integral sum graphs with exactly two saturated nodes, and give an upper bound for the number of edges of a connected integral sum graph with no saturated nodes. We introduce a new method of identification on constructing new connected integral sum graphs from given integral sum graphs with a saturated node. Moreover, we show that every graph is an induced subgraph of a connected integral sum graph. Miscellaneous related results on integral sum graphs are also presented.

2. Preliminaries

Let G_1 and G_2 be two graphs. Suppose $r_1 \in V(G_1)$ is a fixed node of G_1, called the *root* of G_1, and $r_2 \in V(G_2)$ is the root of G_2. We let $(G, r) \equiv (G_1, r_1) \bowtie (G_2, r_2)$ denote the graph G with *root* r, which is obtained from G_1 and G_2 by identifying r_1 and r_2 as one node r. When we do not consider the node r as the root of the obtained graph, we simply denote the graph as $G = (G_1, r_1) \bowtie (G_2, r_2)$. It is clear that $V(G) = (V(G_1) - \{r_1\}) \cup (V(G_2) - \{r_2\}) \cup \{r\}$ and $E(G) = E(G_1) \cup E(G_2)$. For the sake of convenience, we may consider G_1 and G_2 as subgraphs of G and consider r, r_1 and r_2 as the same node. It is also clear that the operation of identification $\bowtie$ is commutative and associative.

Let $G = (V(G), E(G))$ be a graph with node set $V(G)$ and edge set $E(G)$. Let $\overline{G}$ denote the complement of G. Assume that f is a labeling of $V(G)$ with distinct integers. An edge $uv \in E(G) \cup E(\overline{G})$ is said to be *f-proper* if $f(u) + f(v) = f(w)$ for some $w \in V(G)$. Then we immediately have the following fact.

Fact 1. The labeling f is an integral sum labeling of G if and only if all edges of G are f-proper and all edges of $\overline{G}$ are not f-proper.

For an integral sum labeling f of G, the following facts can also be easily seen:

Fact 2. For any non-zero integer m, $m \cdot (f(x))$ also gives an integral sum labeling of G. (We will denote this labeling as mf.)

Fact 3. If $f(v) = 0$ for a node v of G, then v is a saturated node of G.

We use the notation $G^+\{a_1, a_2, \ldots, a_p\}$ to denote an (integral) sum graph with an (integral) sum graph labeling such that the nodes of G are labeled by the integers $a_1, a_2, \ldots, a_p$. It is clear that $G^+\{a_1, a_2, \ldots, a_p\}$ generated by the integers $\{a_1, a_2, \ldots, a_p\}$ is unique up to isomorphism.

3. Main results

Theorem 1. *Let G be an integral sum graph. Then*

(i) *G has at most two saturated vertices unless $G = K_3$,*
(ii) *$G \cong G^+\{1, 0, -1, -2, \ldots, -p+2\}$ if G has exactly two saturated nodes and $|V(G)| = p$.*

Theorem 2. *Let G be a connected integral sum graph with $p > 1$ nodes and q edges. If G has no saturated nodes, then*

$$q \leqslant \left\lfloor \frac{p(3p-2)}{8} \right\rfloor - 2.$$

Theorem 3. *Let (G_i, r_i) be an integral sum graph with a saturated node r_i, for $i = 1, 2, \ldots, n$. Then*

$$G = (G_1, r_1) \bowtie (G_2, r_2) \bowtie \cdots \bowtie (G_n, r_n) \text{ is an integral sum graph.}$$

Theorem 4. *Every graph is an induced subgraph of a connected integral sum graph.*

To prove the theorems, we need the following lemmas.

Lemma 1. *Let G be a nontrivial integral sum graph with exactly one saturated node v. Then for any integral sum labeling f of G, $f(v) = 0$.*

Proof. By contradiction. Assume that there is an integral sum labeling f of G such that $f(v) \neq 0$. Clearly we may assume $f(v) > 0$. (Otherwise we may consider the labeling $(-1)f$ instead.)

Let u be the node with the largest label in $V - \{v\}$. Then $f(u) + f(v) > f(u)$, which implies that $f(u) + f(v) = f(v)$. Therefore $f(u) = 0$ so that u is another saturated node. It contradicts that G has exactly one saturated node. □

Lemma 2. *An integral sum graph $G \neq K_3$ has at most two saturated nodes.*

Proof. Clearly we only need to prove for integral sum graphs G with $|V(G)| > 3$. Assume G has more than one saturated node. We shall show that G has exactly two saturated nodes.

It is easily seen that there is a saturated node v of G and an integral sum graph labeling f such that $f(v) > 0$. Let u be the node with the largest label in $V - \{v\}$. Then $f(v) + f(u) > f(u)$, which implies $f(v) + f(u) = f(v)$. So, $f(u) = 0$, and then $f(x) < 0$ for all $x \in V - \{u, v\}$. Let w be the node with the smallest label in $V - \{u, v\}$. Then w cannot be adjacent to any other node in $V - \{u, v\}$. (Note that $V - \{u, v, w\}$ is not empty by the assumption $|V(G)| > 3$.) Therefore, G cannot have saturated nodes other than u and v. □

Lemma 3. *Let G be an integral sum graph with $|V(G)| = p$. If G has exactly two saturated nodes, then $G \cong G^+\{1, 0, -1, -2, \ldots, -p+2\}$.*

Proof. Since G has two saturated nodes, we have $p \geqslant 2$. When $p = 2$, $G = K_2$ and so $G \cong G^+\{1, 0\}$. This shows that it is true for the case $p = 2$. It is not difficult to see $p \neq 3$, since G cannot have exactly two saturated nodes when $p = 3$. So, we only need to consider the case $p > 3$. Let u and v be the two saturated nodes of G. It is easily seen from the proof of Lemma 2 that there is an integral sum graph labeling f of G such that $f(v) > 0$, $f(u) = 0$, and $f(x) < 0$ for all $x \in V - \{u, v\}$.

Denote the nodes in $V - \{u, v\}$ as $x_1, x_2, \ldots, x_{p-2}$ such that $0 > f(x_1) > f(x_2) > \cdots > f(x_{p-2})$. Since v is adjacent to all $x_i's$ and $f(v) > f(v) + f(x_1) > f(v) + f(x_2) > \cdots > f(v) + f(x_{p-2}) > f(x_{p-2})$, we must have $f(v) + f(x_1) = f(u) = 0$, and $f(v) + f(x_i) = f(x_{i-1})$ for all $i = 1, 2, \ldots, p-2$.

Let $f(v) = M$. Then $f(x_i) = -iM$ for all $i = 1, 2, \ldots, p-2$. It is easy to see that under the new integral sum graph labeling $(1/M)f$, G has labels $1, 0, -1, -2, \ldots, -p+2$. Therefore, $G \cong G^+\{1, 0, -1, -2, \ldots, -p+2\}$. □

Lemma 4. *Let G be an integral sum graph with a saturated node v. Then there is an integral sum labeling f of G such that $f(v) = 0$.*

Proof. It is easy to verify the case $|V(G)| \leqslant 3$. So we may assume $|V(G)| > 3$ in the proof. If G has exactly one saturated node, it is already proved in Lemma 1. If G has more than one saturated node, G must have exactly two saturated nodes by Lemma 2. Then from Lemma 3, we see that there is an integral sum labeling f of G such that one of the saturated nodes has 0 as its label. Then by the symmetry of the two saturated nodes, there is an integral sum labeling f of G such that $f(v) = 0$. □

Before we state the next lemma, let us recall that the join of two graphs G_1 and G_2, denoted as $G_1 \vee G_2$, is the graph obtained from the disjoint union of G_1 and G_2 by adding the edges joining every node of G_1 with every node of G_2.

Lemma 5. *For any sum graph G, the join $K_1 \vee G$ is an integral sum graph.*

Proof. Let g be a sum labeling of G. Then $g(x) > 0$ for all $x \in V(G)$. We may define a labeling f of $V(K_1 \vee G)$ as follows:

$$f(x) = \begin{cases} g(x) & \text{if } x \in V(G), \\ 0 & \text{if } x \in V(K_1). \end{cases}$$

It is clear that f is a labeling of $V(K_1 \vee G)$ with distinct non-negative integers. We shall show that f is an integral sum labeling of $K_1 \vee G$. It is easy to see that every $e \in E(K_1 \vee G)$ is f-proper. So, by Fact 1, we only need to show that any edge in $E(\overline{K_1 \vee G})$ is not f-proper. Let $uv \in E(\overline{K_1 \vee G})$. Then, $\{u, v\} \subset V(G)$, and uv is not g-proper. Since $f(u) + f(v) = g(u) + g(v) > 0$, uv is not f-proper either. This completes the proof of Lemma 5. $\square$

Now we are ready to prove the theorems.

Proof of Theorem 1. By Lemmas 2 and 3 directly. $\square$

Proof of Theorem 2. Let f be an integral sum labeling of G with $f(v_i) = a_i$ for $i = 1, 2, \ldots, p$. Without loss of generality, we may assume that $a_1 < a_2 < \cdots < a_p$. Since G is a connected graph without a saturated node, we easily see that $p > 3$ and that $a_i \neq 0$ for all $i = 1, 2, \ldots, p$ by Fact 3. Since any nontrivial connected graph is not a sum graph, we must have $a_1 < 0$ and $a_p > 0$. Let $a_k (1 \leqslant k < p)$ be the largest among all the negative labels. Then

$$a_1 < a_2 < \cdots < a_{k-1} < a_k < 0 < a_{k+1} < a_{k+2} < \cdots < a_p.$$

We may assume $k \geqslant 2$. (Otherwise, we may consider a new integral sum labeling $(-1\ f$ instead.)

For the sake of notational simplicity, from now on we shall rewrite v_{k+j} as u_j for all $j = 1, 2, \ldots, h$ where $h = p - k$. Since the number of $v_i's$ adjacent to v_1 is not greater than 0, and the number of $u_j's$ adjacent to v_1 is not greater than h. We have $\deg(v_1) \leqslant 0 + h$. Similarly, $\deg(v_i) \leqslant (i-1) + h$ for all $i = 1, 2, \ldots, k-1$.

Note that $a_k + a_i \leqslant a_k + a_{k-1} < a_{k-1}$ for $i = 1, 2, \ldots, k-1$. Then the number of $v_i's$ adjacent to v_k is not greater than $k-2$. It is also easy to see that the numer of $u_j's$ adjacent to v_k is not greater than $h-1$. (The reason is that the inequality $a_k < a_k + a_{k+1} < a_{k+1}$ implies v_k is not adjacent to u_1.) Then we have $\deg(v_k) \leqslant (k-2) + (h-1) = (k-1) + h - 2$. It follows that

$$\sum_{i=1}^{k} \deg(v_i) \leqslant \sum_{i=1}^{k} (i-1) + kh - 2 = \frac{k(k-1)}{2} + kh - 2. \tag{1}$$

Now we distinguish two cases depending on $h \geqslant 2$ or $h = 1$.

Case 1. $h \geqslant 2$. Then, as the above, we have

$$\sum_{j=1}^{h} \deg(u_j) \leqslant \frac{h(h-1)}{2} + hk - 2.$$

Therefore,

$$\begin{aligned} q &= \frac{\sum_{i=1}^{k} \deg(v_i) + \sum_{j=1}^{h} \deg(u_j)}{2} \leqslant \frac{k(k-1) + h(h-1)}{4} + kh - 2 \\ &= \frac{(k+h)^2 - 2kh - (k+h)}{4} + kh - 2 = \frac{p^2 - p}{4} + \frac{kh}{2} - 2 \leqslant \frac{p^2 - p}{4} + \frac{p^2}{8} - 2 = \frac{p(3p-2)}{8} - 2, \end{aligned}$$

which implies the desired inequality.

Case 2. $h = 1$. Then, u_1 is not adjacent to v_k since

$$f(v_k) < f(v_k) + f(u_1) = f(v_k) + f(v_{k+1}) < f(v_{k+1}).$$

Thus we have

$$\deg(u_1) \leqslant k - 1. \tag{2}$$

Then it follows from (1) and (2) that

$$\begin{aligned} q &= \frac{\sum_{i=1}^{k} \deg(v_i) + \deg(u_1)}{2} \leqslant \frac{[k(k-1)/2 + k - 2] + (k-1)}{2} \\ &= \frac{k^2 + 3k - 6}{4} = \frac{(p-1)^2 + 3(p-1) - 6}{4} = \frac{p(p+1)}{4} - 2, \end{aligned}$$

which implies the desired inequality since $(p(p+1)/4)-2 \leqslant (p(3p-2)/8)-2$ as $p \geqslant 4$.

This completes the proof of Theorem 2. □

Proof of Theorem 3. Clearly, we only need to prove it for $n=2$, and we may assume the graphs G_1 and G_2 are nontrivial. Let $G=(G_1,r_1)\bowtie(G_2,r_2)$. By Lemma 4, there is an integral sum labeling f_i of G_i such that $f_i(r_i)=0$, for $i=1,2$. Let $M=\mathrm{Max}\{|f_1(v)| : v \in V(G_1)\}$ and let $f_2'=(3M)f_2$. Clearly, f_2' is an integral sum labeling of G_2 such that $f_2'(r_2)=0$, and $\mathrm{Min}\{|f_2'(v)| : v \in V(G_2)-\{r_2\}\} \geqslant 3M$. Then, for $v \in V(G_1)$, and $v' \in V(G_2)$, $f_1(v) \neq f_2'(v')$ unless $v=r_1$ and $v'=r_2$. Thus we may define a labeling f of $V(G)$ as follows:

$$f(x)=\begin{cases} f_1(x) & \text{if } x \in V(G_1),\\ f_2'(x) & \text{if } x \in V(G_2).\end{cases}$$

It is clear that f is a labeling of $V(G)$ with distinct integers. So, by Fact 1, we only need to show that every edge in $E(G)$ is f-proper and any edge in $E(\overline{G})$ is not f-proper.

Since $E(G)=E(G_1)\cup E(G_2)$, $f|_{V(G_1)}=f_1$ and $f|_{V(G_2)}=f_2'$, we immediately see that every $e \in E(G)$ is f-proper.

Now we shall show that any $e \in E(\overline{G})$ is not f-proper by contradiction. Otherwise, suppose that $uv \in E(\overline{G})$ is f-proper, i.e., $f(u)+f(v)=f(w)$ for some $w \in V(G)$. Without loss of generality, we may distinguish the following three cases:

Case 1. $\{u,v\} \subset V(G_1)$.

Then $uv \in E(\overline{G_1})$. So, uv is not f_1-proper. It follows that $w \notin V(G_1)$, i.e., $w \in V(G_2)-\{r_2\}$. Thus we have $f_1(u)+f_1(v)=f_2'(w)$. However, since $w \in V(G_2)-\{r_2\}$, $|f_2'(w)| \geqslant 3M > 2M \geqslant |f_1(u)|+|f_1(v)| \geqslant |f_1(u)+f_1(v)|$. It is a contradiction.

Case 2. $\{u,v\} \subset V(G_2)$.

It can be proved in the same way as in Case 1.

Case 3. $u \in V(G_1)-\{r_1\}$ and $v \in V(G_2)-\{r_2\}$.

If $w \in V(G_1)$, then $f_1(u)+f_2'(v)=f_1(w)$, i.e., $f_1(u)-f_1(w)=f_2'(v)$. However, since $v \in V(G_2)-\{r_2\}$, $|f_2'(v)| \geqslant 3M > 2M \geqslant |f_1(u)|+|f_1(w)| \geqslant |f_1(u)-f_1(w)|$. It is a contradiction.

If $w \in V(G_2)$, we can get a contradiction similarly.

Therefore, $G=(G_1,r_1)\bowtie(G_2,r_2)$ is an integral sum graph. This completes the proof of Theorem 3. □

Proof of Theorem 4. For any graph G, it is well known [5] that $G \cup mK_1$ is a sum graph where m is the edge number of G. By Lemma 5, $K_1 \vee (G \cup mK_1)$ is an integral sum graph with G as an induced subgraph. □

4. Miscellaneous results

Theorem 1 determines the structure of integral sum graphs with two saturated nodes. Theorem 2 gives an upper bound for the number of edges for an integral sum graph with no saturated nodes. For the class of integral sum graphs with exactly one saturated node, it seems difficult to completely characterize their structures. However, note that any graph with a saturated node is the join $K_1 \vee G$ for some graph G. We have the following:

Proposition 1. *The join $K_1 \vee G$ is an integral sum graph if and only if the nodes of G can be given a labeling f with distinct nonzero integers so that $E(G)=\{uv : f(u)+f(v)=f(w)$ for some $w \in V(G)$ or $f(u)=-f(v)\}$.*

Proof. We first show necessity. By Lemma 4, there is an integral sum graph labeling f' such that $f'(x)=0$ for $x \in V(K_1)$. Let $f=f'|_{V(G)}$. Then f is the desired labeling of the nodes of G.

For sufficiency, it suffices to note that an integral sum graph labeling f' of $K_1 \vee G$ can be defined as

$$f'(x)=\begin{cases} 0 & \text{if } x \in V(K_1),\\ f(x) & \text{if } x \in V(G).\end{cases} \qquad \square$$

As pointed out in Lemma 5, the join of K_1 with any sum graph G is in the class of integral sum graphs with exactly one saturated node. The graphs $K_1 \vee C_i$ for $i \neq 3$ (i.e., the wheels W_n with $n \neq 4$) are also shown [7] to be in this class. Moreover, guided by Proposition 1, we find that all the fans also belong to this class, which is the following.

Proposition 2. *The joins $K_1 \vee P_n$ are integral sum graphs for all paths P_n.*

Proof. Let $P_n = a_1 a_2 \cdots a_n$ and $K_1 = a_0$. An integral sum graph labeling f of $K_1 \vee P_n$ can be defined as follows: $f(a_0) = 0$; $f(a_1) = 1$, $f(a_2) = -1$, $f(a_{k+2}) = f(a_k) - f(a_{k+1})$ for $k = 1, 2, \ldots, n-2$. □

For the join $K_2 \vee G$ to be an integral sum graph, we have the following necessary and sufficient condition, which is a corollary of Theorem 1.

Corollary 1. *The join $K_2 \vee G$ is an integral sum graph if and only if $G \cong G^+\{1, 2, \ldots, n\}$ where $n \geqslant 1$.*

Proof. It is easily seen to be true when G has only one node, since $K_2 \vee K_1 = K_3$ is an integral sum graph and $K_1 \cong G^+\{1\}$.

Then we may assume that G is an integral sum graph with $n > 1$ nodes. It is easy to see the sufficiency, since $K_2 \vee G \cong G^+\{-1, 0, 1, 2, \ldots, n\}$. So we only need to prove the necessity.

Assume that $K_2 \vee G$ is an integral sum graph. Note that $K_2 \vee G$ has $p = n + 2 > 3$ nodes in which at least two (those corresponding to the nodes of K_2) are saturated nodes. By Theorem 1, $K_2 \vee G$ has exactly two saturated nodes (which correspond to the nodes in the K_2), and $K_2 \vee G \cong G^+\{1, 0, -1, -2, \ldots, -n\}$ where $\{1, 0\}$ corresponds to the two saturated nodes. It then follows that $G \cong G^+\{-1, -2, \ldots, -n\} \cong G^+\{1, 2, \ldots, n\}$. This proves the necessity, and so the proof of Corollary 1 is complete. □

The next corollary concludes our discussion on the joins $K_n \vee G$ for all n.

Corollary 2. *If $n > 2$, then the join $K_n \vee G$ is not an integral sum graph for any graph G.*

Proof. It is directly from Theorem 1(i), since $K_n \vee G$ has more than 3 nodes in which at least 3 are saturated nodes. □

Corollary 3. *The complete graph K_n is an integral sum graph if and only if $n = 1, 2, 3$.*

Proof. The sufficiency is well known and very easy to verify. The necessity comes from Corollary 2 since $K_n = K_{n-1} \vee K_1$. □

Corollary 4. *Let G be a connected, k-regular, integral sum graph with $p \geqslant 4$ nodes, then*

$$k \leqslant \left\lfloor \frac{3p-2}{4} - \frac{4}{p} \right\rfloor.$$

Proof. By Theorem 1(i), G has no saturated nodes. Note that the edge number q of G is equal to $kp/2$. Then, from the proof of Theorem 2,

$$\frac{kp}{2} \leqslant \frac{p(3p-2)}{8} - 2.$$

It follows that

$$k \leqslant \left\lfloor \frac{3p-2}{4} - \frac{4}{p} \right\rfloor. \qquad \square$$

From Corollary 4, we can easily obtain the following:

Subcorollary 1. (i) *There is no connected, regular, integral sum graph G with $p = 4$ nodes.*

(ii) *The only connected, regular, integral sum graph G with $p = 5$ nodes is the cycle C_5.*

(iii) *Let G be a connected, k-regular, integral sum graph with $p = 6$ nodes. Then either $k = 3$ or $G = C_6$.*

References

[1] Z. Chen, Integral sum graphs from identification, Discrete Math. 181 (1998) 77–90.
[2] M.N. Ellingham, Sum graphs from trees, Ars Combin. 35 (1993) 335–349.

[3] J. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. 5 (1998) #DS6 (revised Jan. 31, 2000).
[4] F. Harary, Graph Theory, Addison-Wesley, Reading, 1969.
[5] F. Harary, Sum graphs and difference graphs, Congr. Numer. 72 (1990) 101–108.
[6] F. Harary, Sum graphs over all the integers, Discrete Math. 124 (1994) 99–105.
[7] A. Sharary, Integral sum graphs from complete graphs cycles and wheels, Arab Gulf Sci. Res. 14 (1) (1996) 1–14.